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The Equivalence and Approximation of Optimal Control Problems*

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INTRODUCTION

Let $x = (x^1, \cdots, x^n)$ denote a point in n dimensional Euclidean space E^n, which describes the state of some physical system having dynamical equations

$$\dot{x}(t) = f(t, x(t), u). \tag{1}$$

The vector u, termed the control, will be allowed to assume values in a compact set U contained in E^r. Several types of "time optimal" problems may now be posed.

First consider initial data $x(0) = x_0$, for (1), and a target set $S \subset [0, \infty) \times E^n$ given. The "open loop" problem is to find that measurable function $u(t)$ with values in U such that the corresponding solution (trajectory) of (1), denoted $\varphi(t; u)$, satisfies $(t_1, \varphi(t_1; u)) \in S$ for minimum value t_1. The "closed loop" or feedback problem does not require initial data, but instead asks that one find that function $u(t, x)$, with values in U, such that the corresponding solution φ of (1) through any point (t_0, x_0) satisfy $(t_1, \varphi(t_1; u)) \in S$ for minimum value of $t_1 \geqslant t_0$. Since the value of the control here depends on a measurement of both time and state, one expects a "stability" under perturbations to exist which need not be present in open loop systems. Therefore, from the viewpoint of applications, the feedback system is desirable.

Conditions for the existence of an optimal control for the open loop problem are given in [*4*, *8*]. On the other hand, in the feedback problem, if

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$u(t, x)$ is discontinuous with respect to x, it is difficult to even answer questions of existence and uniqueness of solutions of the corresponding system (1). In particular, if the maximum principle [7] is used as a constructive method to obtain a candidate for an optimal trajectory, such discontinuities often occur. The main purpose of this paper is to discuss why these discontinuities occur and to show how, in many cases, a given problem admits a natural approximating problem having smooth control.

Specifically, the maximum principle approach to the time optimal problem proceeds by defining $H(t, x, p, u) \equiv p \cdot f(t, x, u) - 1$ (the dot denoting scalar product) and maximizing H with respect to admissible values of u, for each fixed t, x, p. This algebraic maximization hopefully produces a function $u^*(t, x, p)$, often discontinuous. Define $H^*(t, x, p) \equiv H(t, x, p, u^*(t, x, p))$ and form the differential equations

$$\dot{x} = \frac{\partial}{\partial p} H^*(t, x, p), \qquad \dot{p} = -\frac{\partial}{\partial x} H^*(t, x, p). \tag{2}$$

The maximum principle then assures us that if $u(t)$ is an optimal (open loop) control, and $\varphi(t; u)$ the corresponding solution of (1), there exists an absolutely continuous n vector function $p(t)$, not identically zero, such that

$$H^*(t, \varphi(t; u), p(t)) \equiv H(t, \varphi(t; u), p(t), u(t))$$

while φ and p satisfy (2). However, its constructive use consists in solving a two point boundary value problem for the equations (2), which often have their right sides discontinuous in the dependent variables.

For the system (1), let $R(t, x) \equiv \{f(t, x, u) : u \in U\}$. We shall say that the time optimal problem for a system $\dot{x}(t) = g(t, x(t), v)$, $v \in V$, is *equivalent* to that for the system (1) if $\{g(t, x, v) : v \in V\} = R(t, x)$ for all t, x in some domain of interest. For given $\epsilon > 0$ we define the time optimal problem for the system $\dot{x}(t) = h^\epsilon(t, x, (t), v)$, $v \in V(\epsilon)$, to be an ϵ *approximate equivalent* problem to the time optimal problem for (1) if for all (t, x), $\{h^\epsilon(t, x, v) : v \in V(\epsilon)\}$ contains $R(t, x)$ and the Hausdorff distance between these sets is less than ϵ.

Intuitively equivalent problems have the same optimal trajectories (as shown in Theorem 3) while the optimal trajectories of an ϵ approximate equivalent problem will be close (uniformly) to those of the original problem. The main result, Theorem 4, shows that if a time optimal problem satisfies the conditions of Filippov [4] for the existence of an optimal (open loop) control, then for any $\epsilon > 0$ there exists an ϵ approximate equivalent problem for which the algebraic maximization encountered in the maximum principle, yields a C^1 (once continuously differentiable) function $u^*(t, x, p)$. This function can be related to the optimal feedback control, as in [6], via the

Hamilton Jacobi equation, thereby yielding a method of discussing when a C^1 optimal feedback control exists.

An example is given to illustrate the construction of an approximating problem and its use in generating fields of optimal trajectories via the Hamilton Jacobi equation.

1. The Maximization of a Linear Functional on a Strictly Convex Set

Our motivation is to choose approximating problems for which the maximum principle will yield smooth controls. Let $r^*(p)$ be the function which maximizes the linear functional $p \cdot r$ for fixed $p \in E^n - \{0\}$, $r \in R$ a given compact set in E^n. We begin by examining conditions on the set R which will insure that r^* is smooth, since it is a maximization of this type which causes discontinuities in the control. It is convenient for our purposes to denote the scalar product $p \cdot r$ by $F(p, r)$ and this notation will be used throughout this section.

If S is a set contained in E^n a *support hyperplane* is a hyperplane M which lies on one side of S and $S \cap M \neq \phi$, the empty set. A convex set R contained in E^n will be said to be strictly convex if it contains more than one point, and every support hyperplane has at most one point in common with R.

If R is a compact set in E^n its boundary will be denoted by ∂R.

LEMMA 1. *If R is a strictly convex set in E^n, then R has internal (interior) points.* (This result depends on finite dimensionality).

Proof. Let r_0, $r_1 \in R$, $r_0 \neq r_1$, and V_1 be the linear variety of dimension one determined by these points. Let M_1 be any hyperplane containing V_1. Since M_1 contains two points of R it is not a support plane and there exists a point $r_2 \in R$, $r_2 \notin M_1$. Let V_2 be the linear variety determined by r_0, r_1 and r_2; V_2 has dimension two. Let M_2 be a hyperplane containing V_2. Again there is a point $r_3 \in R$, $r_3 \notin M_2$. We continue inductively getting at the $(n - 1)$st step a linear variety V_{n-1} of dimension $(n - 1)$ determined by the points $r_0, \cdots, r_{n-1}$. Then there exists a unique hyperplane M_{n-1} containing V_{n-1}, and again a point $r_n \in R$, $r_n \notin M_{n-1}$. Since R is convex it contains the convex hull of the set of points $r_0, \cdots, r_n$; and since the vectors $r_1 - r_0$, $r_2 - r_0, \cdots, r_n - r_0$ are linearly independent, they determine an n cell which has non void interior.

LEMMA 2. *Let R be a strictly convex, compact set in E^n. Then for any fixed $p \in E^n - \{0\}$, the function $F(p, \cdot)$ attains its maximum value at a unique point $r^*(p) = r_0 \in \partial R$.*

Proof. For any fixed p, $F(p, \cdot)$ is a continuous function on the compact set R and hence attains its maximum there. Suppose the maximum is attained at an interior point $r_0 \in R$. Let $N(r_0)$ be a neighborhood of r_0 contained in R. Then $F(p, r_0)$ is an interior point of the real interval$F(p, N(r_0))$ contradicting the fact that F attains its maximum at r_0.

To show uniqueness, assume $F(p, \cdot)$ attains its maximum at r_0, while $r_1 \neq r_0$ belongs to R and $F(p, r_1) = F(p, r_0)$. Then $F(p, \cdot)$ is constant on the linear variety V of dimension one determined by r_0 and r_1. But as shown above, no point on this variety can belong to the (nonvoid) interior of R. Therefore by Theorem 3.6E of [9], there exists a closed hyperplane M containing V such that the interior of R lies strictly on one side of M. It follows that M is a support plane for R containing more than one point of R, a contradiction to strict convexity.

THEOREM 1. *Let R be a strictly convex, compact set in E^n. Then the function $r^*(p)$ (shown to be well defined in Lemma 2) is continuous.*

Proof. Suppose $p_n \to p \neq 0$. Since R is compact, some subsequence of the sequence $r^*(p_n)$ converges to a point $r_1 \in R$; assume it is the original sequence. We suppose $r^*(p) = r_2 \neq r_1$ and seek a contradiction. From the definition of r^*, $F(p, r_2) > F(p, r_1)$; let $F(p, r_2) - F(p, r_1) = \delta > 0$. Since F is continuous, there exists an $N > 0$ such that $| F(p_n, r_2) - F(p, r_2) | < \delta/4$ and $| F(p, r_1) - F(p_n, r^*(p_n)) | < \delta/4$ for $n \geqslant N$. Then

$$F(p_n, r_2) - F(p_n, r^*(p_n)) \equiv [F(p, r_2) - F(p, r_1)] + [F(p_n, r_2) - F(p, r_2)] + [F(p, r_1) - F(p_n, r^*(p_n))] > \frac{\delta}{2}$$

or

$$F(p_n, r^*(p_n)) < F(p_n, r_2) \qquad \text{for} \qquad n \geqslant N,$$

a contradiction to the definition of $r^*(p_n)$.

We next examine when the function $r^*(p)$ is C^1. For $y \in E^n$, the notation $| y |$ will be used to denote the Euclidean length of y.

LEMMA 3. *Let R be a strictly convex, compact set in E^n which has a unique outward unit normal $n(r)$ at each point $r \in \partial R$. Then for fixed $p \in E^n - \{0\}$, $F(p, \cdot)$ achieves its maximum at the unique point $r_0 \in \partial R$ such that $n(r_0) = p/| p |$.*

Proof. Assume without loss of generality that zero is an interior point of R. For $x \in E^n$, let $I(x) = \{a : a > 0, a^{-1}x \in R\}$ and define $\rho(x) = \inf. a \in I(x)$; $\rho(x)$ is called the support function of R, or also the Minkowski functional.

We note that if $r_0 \in \partial R$ and y is any vector, then for a real scalar $\alpha > 0$,

$$\frac{y + r_0}{\rho(\alpha y + r_0)} \in \partial R$$

and for α sufficiently small, is in a neighborhood of r_0.

From Lemma 2, we know $F(p, \cdot)$ achieves its maximum at a unique point on ∂R; let r_0 be the point. Let

$$g(y, r_0) = \lim_{\alpha_+ \to 0} \frac{1}{\alpha} \left\{ \frac{y + r_0}{\rho(\alpha y + r_0)} - r_0 \right\}.$$

Since ∂R has a unique outward normal at each point, $g(y, r_0) = -g(-y, r_0)$, while from the condition $F(p, r_0) \geqslant F(p, r)$ for all $r \in \partial R$ in a neighborhood of r_0, it follows that $F(p, g(y, r_0)) \leqslant 0$ for all y. Assuming there exists y such that $F(p, g(y, r_0)) < 0$ implies $F(p, g(-y, r_0)) > 0$, a contradiction. Thus $F(p, g(y, r_0)) = 0$ for all y, or a necessary condition that r_0 presents $F(p, \cdot)$ a maximum is that p be orthogonal to the support hyperplane at r_0. Since R is strictly convex, it is easily shown that there are exactly two points which satisfy this necessary condition, one with outward normal $p/|p|$ giving F a maximum, the other with normal $-p/|p|$ which gives F a minimum.

We shall say that a strictly convex, compact set R in E^n has a *smooth boundary* if there exists a unique outward unit normal $n(r) \in C^1$ defined on ∂R. (Actually, we consider n as a restriction of a C^1 function in a neighborhood of $r \in \partial R$; see, for example, [*1*], p. 27.)

THEOREM 2. *If R is a compact set in E^n with smooth boundary having positive Gaussian curvature at all points, then $r^*(p) \in C^1$.*

Proof. Since it is assumed that the unit normal to ∂R is of class C^1, the Gaussian curvature is a continuous positive function on ∂R. But ∂R is compact, thus the Gaussian curvature is bounded away from zero. From Theorem 5.4, which readily extends by induction to $n \geqslant 3$, and Corollary 5.5 [*3*, p. 35], it follows that R is strictly convex.

From Lemma 3, $r^*(p)$ satisfies $n(r^*(p)) \equiv p/|p|$. Let $r_0 = r^*(p_0)$ be an arbitrary point on ∂R.

The method will be to utilize the implicit function theorem on a relation of the form $g(r, p) \equiv n(r) - p/|p|$.

Let $\zeta^1, \cdots, \zeta^{n-1}$ be a local coordinate system for a neighborhood of r_0 on ∂R. Then the inclusion map from $\partial R \to E^n$ determines n smooth functions $x_1(\zeta^1, \cdots, \zeta^{n-1}), \cdots, x_n(\zeta^1, \cdots, \zeta^{n-1})$ or briefly $x(\zeta)$. Assume $x(0) = r_0$ and let V_1 be a measurable neighborhood of zero in the local coordinate system.

Let S^{n-1} be the unit $(n-1)$ sphere; we consider $n(\cdot)$: $\partial R \to S^{n-1}$. Define $\theta(\cdot) : V_1 \to S^{n-1}$ by $n(x(\zeta)) = \theta(\zeta)$. Thus $n \in C^1 \Rightarrow \theta \in C^1$.

Let $\varphi = \varphi(p) = p/|p|$, $p \in E^n - \{0\}$; then $\varphi \in C^1$. Our approach will be to utilize the implicit function theorem on the relation $G(\zeta, \varphi) = \theta(\zeta) - \varphi$.

We note that $G \in C^1$, and if $\varphi_0 = \varphi(p_0)$ then $G(0, \varphi_0) = 0$. Also $G_\zeta(0, \varphi_0) = \theta_\zeta(0)$. It must be shown that $\det(\theta_\zeta(0)) \neq 0$.

From differential geometry we recall that as ζ varies in V_1, $x(\zeta)$ traces out a region V_2 on ∂R while the normal $\theta(\zeta)$ traces out a region V_3 on the surface of the unit sphere. Let $K(\zeta)$ denote the Gaussian curvature of ∂R at $x(\zeta)$, and A_3 the "area" of V_3. Then

$$A_3 = \int_{V_1} K(\zeta)\, d\zeta.$$

But

$$\int_{V_1} \det\left(\frac{\partial\theta(\zeta)}{\partial\zeta}\right) d\zeta = A_3 .$$

Since V_1 is arbitrary (but measurable) and $\theta \in C^1$, this implies

$$\det\left(\frac{\partial\theta(\zeta)}{\partial\zeta}\right) = K(\zeta).$$

By assumption K is positive at all points of ∂R, hence $\det(\theta_\zeta(0)) \neq 0$. The implicit function theorem now gives the existence of a C^1 function $\zeta(\varphi)$ such that $G(\zeta(\varphi), \varphi) \equiv 0$.

Then $r^*(p) = x(\zeta(\varphi(p))) \in C^1$.

The following is an example of a strictly convex set R with smooth boundary and a point at which the Gaussian curvature K is zero, for which $r^*(p)$ is not C^1.

Let part of the boundary of $R \subset E^2$ consist of the curve $y = x^4$, $-1 \leqslant x \leqslant 1$, the rest so as to make R strictly convex and with smooth boundary. We restrict our attention to the defined part of the boundary, in particular to the point $(0, 0)$ at which K is zero.

The outward normal is given by $(4x^3, -1)$. Let $p = (p_1, p_2)$ have p_2 negative and p_1 near zero. To compute $r^*(p) \equiv (x^*(p), y^*(p))$ we compute the point on the curve $y = x^4$ where the normal has direction numbers $(-p_1/p_2, -1)$. This gives $x^*(p) = (-p_1/4p_2)^{1/3}$, $y^*(p) = (-p_1/p_2)^{4/3}$, and $\partial x^*(p)/\partial p_1$ is seen not to be continuous at $p_1 = 0$.

2. Approximation of Optimal Trajectories

The Time Optimal Problems

Consider the system (1), with U a compact set, and initial data $x(t_0) = x_0$. Let S be a smooth (C^2) manifold in the $(n+1)$-dimensional (t, x) space with the property that for any t_2, t_3, $\{(t, x) \in S : t_2 \leqslant t \leqslant t_3\}$ is compact in E^{n+1}. The problem is to find a measurable function $u = u(t)$ having values in U, such that the solution of the initial value problem for (1) with $u = u(t)$, intersects the target S in minimum time; i.e., is an optimal trajectory.

We next give the conditions of Filippov [*4*], which insure the existence of an optimal (open loop) control, and optimal trajectory for the time optimal problem.

Existence Conditions

(3) $f(t, x, u)$ is continuous in all variables t, x and u, and is continuously differentiable with respect to x.

(4) $x \cdot f(t, x, u) \leqslant C(|x|^2 + 1)$ for all t, x, u.

(5) $R(t, x) \equiv \{f(t, x, u) : u \in U\}$ is convex for every t, x.

(6) There exists at least one measurable function $u(t)$ with values in U, such that the corresponding solution of the initial value, problem for (1) attains the target S for some $t_1 \geqslant t_0$.

Equivalence of Problems

Let the same time optimal problem, as posed for (1), also be posed for the system

$$\dot{x}(t) = g(t, x(t), v(t)), \qquad v(t) \in V, \qquad \text{a compact set,} \tag{7}$$

where g satisfies condition (3). Let

$$Q(t, x) \equiv \{g(t, x, v) : v \in V\}.$$

THEOREM 3. *Assume the existence conditions are satisfied for the time optimal problem for the system (1). Let $\varphi(\cdot; u^*)$ denote the optimal trajectory and u^* the optimal control. Then if $Q(t, x) = R(t, x)$ for all (t, x), $\varphi(\cdot; u^*)$ is an optimal trajectory for the time optimal problem for the system (7) and there exists a measurable function $v^*(t)$ with values in V such that*

$$\dot{\varphi}(t; u^*) = g(t, \varphi(t; u^*), v^*(t))$$

almost everywhere.

Proof. $f(t, \varphi(t; u^*), u^*(t))$ is a measurable function of t, with values (almost everywhere) in $R(t, \varphi(t; u^*))$, therefore in $Q(t, \varphi(t; u^*))$. From Lemma 1 of Filippov [*4*], there exists a measurable function $v^*(t)$ with values in V such that

$$f(t, \varphi(t; u^*), u^*(t)) = g(t, \varphi(t; u^*), v(t))$$

almost everywhere. It follows that

$$\dot{\varphi}(t; u^*) = g(t, \varphi(t; u^*), v^*(t))$$

almost everywhere.

Now if $\varphi(\cdot; u^*)$ were not an optimal trajectory for (7), i.e., $\psi(\cdot; v)$ provides a better time, the same argument shows that $\psi(\cdot; v)$ is a solution of (1) for some measurable control u with values in U, thereby contradicting the assumed optimality of $\varphi(\cdot; u^*)$.

This theorem stresses the fact that in seeking optimal trajectories, it is the set function $R(t, x)$ which is of major importance, not the function $f(t, x, u)$ or the control set U.

When the condition of Theorem 3 are satisfied, *we define the time optimal problem for the system (7) to be equivalent to that for (1).*

If the existence conditions are satisfied for the time optimal problem, from conditions (4) and (6) we can obtain a compact region of (t, x) space to which analysis can be restricted. Indeed for $t_0 \leqslant t \leqslant t_1$ condition (4) implies any solution $x(t)$ of (1) satisfies

$$| x(t) |^2 \leqslant (| x_0 |^2 + 1) \exp (2C \mid t_1 - t_0 \mid).$$

Here $| x(t) |$ stands for the usual Euclidean norm. Henceforth, we denote by $\mathscr{D}$ the compact region of (t, x) space defined by $t_0 \leqslant t \leqslant 2t_1$,

$$| x |^2 \leqslant (| x_0 |^2 + 1) \exp (2C \mid 2t_1 - t_0 \mid).$$

DEFINITION. The *Hausdorff metric topology* for nonempty compact sets in E^n is derived from the following metric: The distance between two nonempty compact sets X and Y in the smallest real number $d = d(X, Y)$ such that X lies in the d neighborhood of Y and Y lies in the d neighborhood of X.

ϵ *Approximate Equivalent Problems*

DEFINITION. For given $\epsilon > 0$ the time optimal problem for the system $\dot{x} = h^\epsilon(t, x, v)$, h^ϵ continuous on $E^1 \times E^n \times V(\epsilon)$, $V(\epsilon)$ compact, is said to be an ϵ approximate equivalent problem to the time optimal problem for (1) if the set

$$R(t, x, \epsilon) \equiv \{h^\epsilon(t, x, v) : v \in V(\epsilon)\} \supset R(t, x)$$

and

$$d(R(t, x, \epsilon), R(t, x)) \leqslant \epsilon \qquad \text{for all} \qquad (t, x) \in \mathscr{D}.$$

The condition $R(t, x, \epsilon) \supset R(t, x)$ assures that points attainable by trajectories of the original problem are attainable by those of the approximate problem.

THEOREM 4. *Assume that the Filippov conditions (3), (4), and (5) are satisfied for the time optimal problem with system equations (1). Then for* *every $\epsilon > 0$ there exists an ϵ approximate equivalent problem with system equations $\dot{x} = h^\epsilon(t, x, v)$, $v \in V(\epsilon)$ which satisfies the following properties.*

(a) *The control set $V(\epsilon)$ can be taken to be the unit ball of E^n, which we denote B^n.*

(b) *h^ϵ is a C^∞ function on $\mathscr{D} \times B^n$, while for each $(t, x) \in \mathscr{D}$, $h^\epsilon(t, x, \cdot)$ is one-one on $B \to E^n$.*

(c) *The set $R(t, x, \epsilon) \equiv \{h^\epsilon(t, x, v) : v \in B^n\}$ has smooth boundary having positive Gaussian curvature.*

(d) *The (single valued) function $v^*(t, x, p)$ with values in B^n which maximizes*

$$H(t, x, p, v; \epsilon) = p \cdot h^\epsilon(t, x, v) - 1$$

for each $(t, x) \in \mathscr{D}$, $p \in E^n - \{0\}$, is C^1 in t, x, and p. Actually

$$v^*(t, x, p) \in \partial B^n = S^{n-1},$$

the $(n - 1)$ sphere.

The proof will proceed by obtaining a simplicial approximation to $\mathscr{D}$ in which the diameters of the simplexes are sufficiently small. Fore each vertex (t_i, x_i) of a simplex, we approximate the convex set $R(t_i, x_i)$ by a strictly convex set $Q(t_i, x_i, \epsilon)$ having positive Gaussian curvature. A vector function $g^\epsilon(t_i, x_i; \cdot)$ is then constructed so that

$$Q(t_i, x_i; \epsilon) = (g^\epsilon(t_i, x_i; v) : v \in B^n\},$$

and by use of g^ϵ, the set function Q is extended continuously to all of $\mathscr{D}$ in such a manner that for each $(t, x) \in \mathscr{D}$, $Q(t, x; \epsilon)$ has smooth boundary with positive Gaussian curvature. The desired function h^ϵ is then obtained by smoothing the function g^ϵ in the variables (t, x) via the Friedrichs mollifier technique.

Proof. $R(t, x)$ is continuous, in the Hausdorff metric topology, on the compact set $\mathscr{D}$. For any $\epsilon > 0$ let $\delta > 0$ be such that $d(R(t, x), R(t', x')) < \epsilon/8$ whenever $|(t, x) - (t', x')| < \delta$. Let σ_g^{n+1} be any bounded geometric simplex which contains $\mathscr{D}$, and K_g be the geometric complex consisting of this single simplex. By barycentric subdivision K_g can be subdivided into a geometric complex K_g' consisting of a family of geometric simplexes $\{\bar{\sigma}_g^{n+1}\}$, each having diameter less than δ.

Each point $(t, x) \in \mathscr{D}$ has a unique representation of the form

$$(t, x) = \sum_{i=1}^{n+2} \alpha_i(t_i, x_i)$$

with $0 \leqslant \alpha_i \leqslant 1$, $\Sigma\alpha_i = 1$; where the $(n+2)$ points (t_i, x_i) are the vertices of the geometric simplex from the family $\{\bar{\sigma}_g^{n+1}\}$ to which the point (t, x) belongs. Without loss of generality we can now consider the union of the members of $\{\bar{\sigma}_g^{n+1}\}$ which have all vertices in $\mathscr{D}$ as a new domain of interest; call this domain again $\mathscr{D}$.

Let (t_i, x_i) be an arbitrary vertex in $\mathscr{D}$. Then $R(t_i, x_i)$ is convex. Let $\eta(R(t_i, x_i), \epsilon/4)$ be a convex $\epsilon/4$ neighborhood of $R(t_i, x_i)$. From [2], p. 38, there exists a strictly convex set $Q(t_i, x_i, \epsilon)$ containing $\eta(R(t_i, x_i), \epsilon/4)$; having an analytic boundary with positive Gaussian curvature, and such that

$$d\left(Q(t_i, x_i, \epsilon), \eta\left(R(t_i, x_i), \frac{\epsilon}{4}\right)\right) < \frac{\epsilon}{4}.$$

For each $(t_i, x_i) \in \mathscr{D}$ we construct a corresponding set $Q(t_i, x_i, \epsilon)$ as above. We next proceed to define a set valued function $Q(t, x, \epsilon)$ on all of $\mathscr{D}$.

It can be assumed without loss of generality that $0 \in R(t, x)$ for all $(t, x) \in \mathscr{D}$. Indeed if this were not so, one could choose a point $u_0 \in U$ and construct new sets

$$S(t, x) \equiv \{f(t, x, u) - f(t, x, u_0) : u \in U\}$$

which satisfy this property.

Let B^n be the unit ball in E^n; S^{n-1} its surface and $v^1, \cdots, v^{n-1}$ a coordinate system on S^{n-1} while v^n measures distance from the origin. Then a ray from the origin through $(v^1, v^2, \cdots, v^{n-1}, 1)$ strikes $\partial Q(t_i, x_i, \epsilon)$ in a unique point which we denote $g^\epsilon(t_i, x_i, v^1, \cdots, v^{n-1}, 1)$. This defines $g^\epsilon(t_i, x_i, \cdot)$ on S^{n-1}; to extend it to B^n let $v = (v^1, \cdots, v^n) \in B^n$. Define $g^\epsilon(t_i, x_i, v)$ as that point in $Q(t_i, x_i, \epsilon)$ which lies on the ray through the origin and $(v^1, \cdots, v^{n-1}, 1)$ and is such that

$$\frac{|g^\epsilon(t, x, v)|}{|g^\epsilon(t, x, v^1, \cdots, v^{n-1}, 1)|} = v^n.$$

Then

$$g^\epsilon(t_i, x_i, \cdot) : B^n \to Q(t_i, x_i, \epsilon)$$

in a one to one fashion. We will define $Q(t, x, \epsilon)$ on all of $\mathscr{D}$ by extending the definition of g^ϵ, to all $(t, x) \in \mathscr{D}$.

Assume $(t, x) \in \mathscr{D}$. Let $(t, x) = \sum_{i=1}^{n+2} \alpha_i(t_i, x_i)$ be the unique representation of (t, x) in terms of the vertices of the geometric simplex of K_g' to which it belongs. Define

$$g^\epsilon(t, x, v) = \sum_{i=1}^{n+2} \alpha_i g^\epsilon(t_i, x_i, v), \qquad v \in B^n.$$

Then if

$$Q(t, x, \epsilon) = \{g^{\epsilon}(t, x, v) : v \in B^n\}$$

it follows that:

(i) $\eta(R(t, x), \epsilon/8) \subset Q(t, x, \epsilon)$. Indeed, from the choice of δ,

$$\eta\left(R(t, x), \frac{\epsilon}{8}\right) \subset \eta\left(R(t_i, x_i), \frac{\epsilon}{4}\right) \subset Q(t_i, x_i, \epsilon)$$

for all vertices (t_i, x_i) of the simplex in which (t, x) is contained. But

$$Q(t, x, \in) = \sum \alpha_i Q(t_i, x_i, \epsilon).$$

Thus if a point is in $\eta(R(t, x), \epsilon/8)$ it is in $Q(t, x, \epsilon)$.

(ii) $d(Q(t, x, \epsilon), R(t, x)) < 3\epsilon/4$. To show this one notes that

$$R(t_i, x_i) \subset \eta\left(R(t_j, x_j), \frac{\epsilon}{4}\right) \subset Q(t_j, x_j, \epsilon)$$

for all $i, j = 1. 2, \cdots, n + 2$. Therefore

$$\begin{aligned} d(R(t, x), Q(t, x, \epsilon) &\leqslant d(R(t, x), R(t_i, x_i)) + d\left(R(t_i, x_i), \sum_j \alpha_i Q(t_j, x_j, \epsilon)\right) \\ &\leqslant \frac{\epsilon}{8} + \max_j \,[d(R(t_i, x_i), Q(t_j, x_j, \epsilon)] \\ &\leqslant \frac{\epsilon}{8} + \max_j \,[d(R(t_i, x_i), R(t_j, x_j)) \\ &\qquad + d(R(t_j, x_j), Q(t_j, x_j, \epsilon))] \leqslant \frac{\epsilon}{4}. \end{aligned}$$

(iii) $Q(t, x, \epsilon)$ is strictly convex, with smooth boundary having positive Gaussian curvature, for each (t, x). Indeed if $K(t, x, v^1, \cdots, v^{n-1})$ is Gaussian curvature at the point $g(t, x, v^1, \cdots, v^{n-1}, 1) \in \partial R(t, x, \epsilon)$, then

$$K(t, x, v^1, \cdots, v^{n-1}) = \sum_{i=1}^{n+2} \alpha_i K(t_i, x_i, v^1, \cdots, v^{n-1}).$$

(iv) From the construction, $g^{\epsilon}(t, x, v)$ is analytic in v for fixed (t, x) and continuous in (t, x) for fixed v.

Combining the results of (i) and (ii) shows that for $(t, x) \in \mathscr{D}$,

$$\eta\left(R(t, x), \frac{\epsilon}{8}\right) \subset Q(t, x, \epsilon) \subset \eta\left(R(t, x), \frac{3\epsilon}{4}\right).$$

It will next be shown that using $g^\epsilon(t, x, v)$ one can construct a mapping $h^\epsilon(t, x, v)$ on $\mathscr{D} \times B^n \to E^n$ such that if

$$R(t, x, \epsilon) = \{h^\epsilon(t, x, v) : v \in B^n\},$$

then $R(t, x, \epsilon)$ is a strictly convex, compact set containing $R(t, x)$; $d(R(t, x, \epsilon), R(t, x)) < \epsilon$; $\partial R(t, x, \epsilon)$ is smooth with positive Gaussian curvature, and if $n(t, x, h^\epsilon(t, x, v^1, \cdots, v^{n-1}, 1)$ is a unit normal to $\partial R(t, x, \epsilon)$ at $h^\epsilon(t, x, v^1, \cdots, v^{n-1}, 1)$ then it is a C^1 function of all arguments.

For simplicity of notation let $y = (t, x)$ denote a point in $\mathscr{D}$, and let $S^k(y - \bar{y})$ be a mollifier function; see [5]. As an example one could choose

$$S^k(y - \bar{y}) = \left(\frac{k}{4\pi}\right)^{(n+1)/2} \exp\left\{-\frac{k}{4}\left[\sum_{i=1}^{n+1}(y^i - \bar{y}^i)^2\right]\right\}.$$

Extend $g^\epsilon(y, v)$ as the zero function for y in the complement of $\mathscr{D}$.

Define

$$h^k(y, v) = \int_{E^{n+1}} S^k(y - \bar{y})\, g^\epsilon(\bar{y}, v)\, d\bar{y}.$$

Then for every integer $k > 0$, h^k is an analytic function, while h^k and its derivatives with respect to v tend uniformly to g^ϵ and its derivatives with respect to v.

Let

$$R^k(t, x, \epsilon) \equiv \{h^k(t, x, v) : v \in B^n\}.$$

Since the Gaussian curvature to $\partial Q(t, x, \epsilon)$ is given as a multilinear combination of the derivatives $g^\epsilon{}_{v^i}\,(t, x, v^1, \cdots, v^{n-1}, 1)$ while the curvature of $\partial R^k(t, x, \epsilon)$ is given by the same multilinear combination of the derivatives $h^k_{v^i}(t, x, v^1, \cdots, v^{n-1})$; one can choose k sufficiently large so that $\partial H^k(t, x, \epsilon)$ has positive Gaussian curvature while

$$R(t, x) \subset H^k(t, x, \epsilon) \subset \eta(R(t, x), \epsilon).$$

For such a choice of k, define

$$h^\epsilon(t, x, v) = h^k(t, x, v), \qquad R(t, x, \epsilon) \equiv \{h^\epsilon(t, x, v) : v \in B^n\}.$$

From its construction, h^ϵ satisfies conclusions (a), (b), and (c), while a unit normal $n(t, x, h^\epsilon(t, x, v^1, \cdots, v^{n-1}, 1))$ to $\partial R(t, x, \epsilon)$ is a C^1 function of $(t, x, v^1, \cdots, v^{n-1})$.

It remains to show part (d). Using Lemma 3 define $r^*(t, x, p; \epsilon)$ as the unique point on $\partial R(t, x, \epsilon)$ such that $n(t, x, r^*(t, x, p, \epsilon)) = p/|\,p\,|$. It will be shown that r^* is a C^1 function of t, x, and p by a proof similar to that of Theorem 2. Defining $v^*(t, x, p)$ as the unique point on ∂B^n such that

$$h^\epsilon(t, x, v^*(t, x, p)) = r^*(t, x, p, \epsilon)$$

it follows that v^* maximizes $H(t, x, p, v; \epsilon)$ and it will be shown that v^* is a C^1 in t, x, and p.

For fixed (t, x), we have

$$S^{n-1} \xleftrightarrow{h^\epsilon(t, x, v^1, \cdots, b^{n-1}, 1)} \partial R(t, x, \epsilon) \xleftrightarrow{\qquad n(t, x, r) \qquad} S^{n-1}$$

which naturally induces a map $\theta(t, x, v^1, \cdots, v^{n-1})$ from $S^{n-1} \leftrightarrow S^{n-1}$ defined by

$$\theta(t, x, v^1, \cdots, v^{n-1}) \equiv n(t, x, h^\epsilon(t, x, v^1, \cdots, v^{n-1}, 1)).$$

Since we are only interested in $\partial B^n = S^{n-1}$, no confusion should occur if for the remainder of this argument we let $v = (v^1, \cdots, v^{n-1}) \in S^{n-1}$ and, therefore, write $\theta(t, x, v)$. This will be done.

Let $\varphi = \varphi(p) = p/|p|$, $p \in E^n - \{0\}$ and define

$$G(t, x, v, \varphi) \equiv \theta(t, x, v) - \varphi.$$

We will apply the implicit function theorem to G, which is easily seen to be a C^1 function. For each t_0, x_0, $\varphi_0 = p_0/|p_0|$, there exists a unique point $r_0 = r^*(t_0, x_0, p_0; \epsilon)$ such that if $n(t_0, x_0, r_0) = p_0/|p_0|$ and v_0 is the unique point on S^{n-1} such that $h^\epsilon(t_0, x_0, v_0) = r_0$, then $G(t_0, x_0, v_0, \varphi_0) = 0$. One next notes that $G_v(t_0, x_0, v_0, \varphi_0) = \theta_v(t_0, x_0, v_0)$, and from the definition of θ (see also the proof of Theorem 2) det $[\theta_v(t_0, x_0, v_0)]$ is the Gaussian curvature at $r_0 \in \partial R(t, x, \epsilon)$ which is positive. The implicit function theorem yields the existence of a C^1 function $v(t, x, \varphi)$ such that $G(t, x, v(t, x, \varphi), \varphi) \equiv 0$ in a neighborhood of the arbitrary point t_0, x_0, φ_0. Then

$$r^*(t, x, p; \epsilon) = h^\epsilon(t, x, v(t, x, \varphi(p))) \in C^1, \quad \text{while} \quad v^*(t, x, p) \equiv v(t, x, \varphi(p)))$$

is also C^1.

The Relation of Trajectories of the Approximating Problem to Those of the Time Optimal Problem

We assume the system (1) satisfies the Filippov existence conditions (3), (4), (5), and (6), with t_1 a time in which the target set S is attainable. For any $\epsilon > 0$ let $h^\epsilon(t, x, v)$, $v \in V(\epsilon)$, be an ϵ approximate equivalent problem (not necessarily having the special properties shown to exist in Theorem 4). From condition (6) and the relation $R(t, x, \epsilon) \supset R(t, x)$, it readily follows that for every $\epsilon > 0$ there exists at least one measurable function v with values in $V(\epsilon)$ such that the corresponding trajectory $\varphi^{\epsilon}(\cdot; v)$ of the ϵ approximate problem, attains the target S.

It will next be shown that when dealing with the approximate problem, analysis can again be restricted to a compact set. Indeed any vector $h^{\epsilon}(t, x, v)$ can be written as $f(t, x, u) + \alpha(t, x)$ where $| \alpha(t, x) | \leqslant \epsilon$. Then for any trajectory $x(t)$ of the approximate problem

$$\begin{aligned} \tfrac{1}{2} \frac{d}{dt} | x(t) |^2 &= x(t) \cdot h^{\epsilon}(t, x, (t), v(t)) \\ &= x(t) \cdot f(t, x(t), u(t)) + x(t) \cdot \alpha(t, x(t)) \\ &\leqslant C(1 + | x(t) |^2) + \epsilon \, | x(t) |. \end{aligned}$$

$$\frac{d}{dt} \ln (1 + | x(t) |^2) \leqslant 2C + \frac{2\epsilon \, | x(t) |}{1 + | x(t) |^2} \leqslant 2(C + \epsilon),$$

$$| x(t) |^2 \leqslant (1 + | x_0 |^2) \exp [2(C + \epsilon)(2t_1 - t_0)].$$

Define $\mathscr{D}^{\epsilon}$ to be the compact region in E^{n+1} dimensional (t, x) space so that

$$| x |^2 \leqslant (1 + | x_0 |^2) \exp. [2(C + \epsilon)(2t_1 - t_0)], \qquad t_0 \leqslant t \leqslant 2t_1 .$$

THEOREM 5. *Consider a sequence $\{\epsilon(k)\}$ with $\epsilon(k) > 0$, $\epsilon(k) \to 0$ and let $\varphi^{\epsilon(k)}$ denote the time optimal trajectory (assumed to exist) for the $\epsilon(k)$ approximate problem. Then $\{\varphi^{\epsilon(k)}\}$ is an equicontinuous family on the interval $[t_0, t_1]$. It has a uniformly convergent subsequence which converges to a function φ^* having the following properties.*

(i) *φ^* is absolutely continuous.*

(ii) *There exists a measurable function u^* with values in U such that $\dot{\varphi}^*(t) = f(t, \varphi^*(t), u^*(t))$ almost everywhere.*

(iii) *There exists a smallest $t^* \geqslant t_0$ such that $\varphi^*(t^*) \in S$.*

(iv) *φ^* is a time optimal trajectory for the system* (1).

Proof. We shall prove the conclusions in the order that they are stated. Without loss of generality, assume that

$$R(t, x, \epsilon(1)) \supset R(t, x, \epsilon(2)) \supset \cdots R(t, x).$$

Therefore analysis can be restricted to the compact region $\mathscr{D}^{\epsilon(1)}$. Our first goal is to show that there is a constant N independent of $\epsilon(k)$ such that $\varphi^{\epsilon(k)}$ is Lipschitz continuous with Lipschitz constant N. To accomplish this, for a compact set R in E^n let $\rho(R)$ denote $\max_{r \in R} | r |$. For fixed $\epsilon(1)$,

$R(t, x, \epsilon(1))$ is a continuous set valued function (in the Hausdorff metric topology) on the compact set $\mathscr{D}^{\epsilon(1)}$ and therefore the composite map $\rho(R(t, x, \epsilon(1)))$ is a continuous real valued function on $\mathscr{D}^{\epsilon(1)}$, hence bounded. Let N be its bound. Denote by h^k an $\epsilon(k)$ approximating system. It follows that $| h^{\epsilon(k)}(t, x, v) | \leqslant N$ for all $\epsilon(k)$ and any trajectory $\varphi^{\epsilon(k)}$ is Lipschitz continuous with Lipschitz constant N. Thus $\{\varphi^{\epsilon(k)}\}$ is equicontinuous and has a subsequence which converges uniformly to a Lipschitz continuous function φ^*, which is therefore absolutely continuous. We will not distinguish between $\{\varphi^{\epsilon(k)}\}$ and its convergent subsequence.

(ii) We next show that for almost all $t \in [t_0, t_1]$, $\dot{\varphi}^*(t) \in R(t, \varphi^*(t))$. Since the set function $R(t, x)$ is continuous in the Hausdorff metric topology (a consequence of the continuity of f), for any $\nu > 0$ let $R_\nu(t, x)$ be a closed convex ν-neighborhood of $R(t, x)$. Then $R_\nu(t, x)$ is also a continuous set function.

Since

$$\dot{\varphi}^{\epsilon(k)}(t) \in R(t, \varphi^{\epsilon(k)}(t), \epsilon(k)) \qquad \text{and} \qquad R(t, x, \epsilon(k)) \to R(t, x)$$

in the Hausdorff metric topology, there exists an N such that for all $n \geqslant N$, $\dot{\varphi}^{\epsilon(k)}(t) \in R_\nu(t, \varphi^*(t))$. Filippov's proof of Theorem 1, [4], now applies to show that for almost all t, $\dot{\varphi}^*(t) \in R_\nu(t, \varphi^*(t))$. But $R(t, x)$ is closed and ν arbitrarily small, hence $\dot{\varphi}^*(t) \in R(t, \varphi^*(t))$ for almost all t.

From the lemma of Filippov [4], we then obtain the existence of a measurable control u^* with values in U, such that for almost all $t \in [t_0, t_1]$,

$$\dot{\varphi}^*(t) = f(t, \varphi^*(t), u^*(t)).$$

(iii) Let $t_{\epsilon(k)} > t_0$ denote the optimal time for the $\epsilon(k)$ approximate problem. Since $R(t, x, \epsilon(1)) \supset R(t, x, \epsilon(2)) \supset \cdots$ it follows that $\{t_{\epsilon(k)}\}$ is a monotone non-decreasing sequence of reals bounded above by t_1. Let t^* be its limit. Now $\varphi^{\epsilon(k)}(t_{\epsilon(k)}) \in S$ for each k, and $\{(t, x) \in S : t_0 \leqslant t \leqslant t_1\}$ is a compact in E^{n+1}, thus $\varphi^{\epsilon(k)}(t_{\epsilon(k)}) \to \varphi^*(t^*) \in S$.

(iv) Suppose φ^* is not a time optimal trajectory for the system (1). Then there exists a measurable control u with values in U and corresponding trajectory $\varphi(\cdot; u)$ such that $\varphi(t_0; u) = x_0$, $\varphi(t_3; u) \in S$ and $t_3 < t^*$. This implies that for k sufficiently large, $t_3 < t_{\epsilon(k)}$; but $\varphi(\cdot; u)$ is an admissible trajectory to all ϵ approximate problems. This contradicts the optimality of $\varphi^{\epsilon(k)}$.

This theorem essentially tells us that for sufficiently small ϵ, the optimal trajectories of the ϵ approximate problem are uniformly close to optimal trajectories of the original problem.

The Construction of Approximating Problems when the Control Appears Linearly

Theorem 4 gives conditions for the existence of an ϵ equivalent approximate problem which has the unit ball B^n as the set of values which the control can assume. However, the functional form of the approximating system is allowed to vary with ϵ.

In this section we consider a system of the form

$$\dot{x}(t) = g(t, x(t)) + H(t, x(t))\, u(t), \tag{8}$$

$u(t) \in U$, a compact convex set in E^r with $1 \leqslant r \leqslant n$; H an $n \times r$ matrix valued C^2 function; while g is a C^2, n vector valued function. For such systems it is possible to provide a simple construction for ϵ approximate problems.

Since, for the approximate problem, one desires $R(t, x, \epsilon)$ to be strictly convex, and Lemma 1 shows this implies a nonvoid interior, one is led to extend H to an $n \times n$ matrix valued function and approximate the control set by a compact set $V(\epsilon)$ which contains U. Furthermore, $V(\epsilon)$ should have a nonvoid n-dimensional interior, a smooth boundary with positive Gaussian curvature, and be such that in the Hausdorff metric topology,

$$\lim_{\epsilon \to 0} V(\epsilon) = U.$$

The method of construction and the application to approximating problems will be demonstrated in a two-dimensional example; its generalization to higher dimensions being immediate.

Example 1 (Bushaw control problem).

Consider the time optimal problem for the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + u$$

with arbitrary initial data $x(0) = x_0$, and target

$$S = \{(t, x_1, x_2) : x_1 = 0, x_2 = 0\}.$$

The control u is to satisfy $-1 \leqslant u(t) \leqslant 1$, i.e., $U = [-1, 1]$.

As an ϵ approximate problem we take the system

$$\dot{x}_1 = x_2 + v_1, \qquad \dot{x}_2 = -x_1 + v_2 \tag{9}$$

with the same initial data and target, but with

$$V(\epsilon) = \{v \in E^2 : v_1^2 + \epsilon^2 v_2^2 \leqslant \epsilon^2\},$$

i.e., an ellipse with semimajor axis 1 and semiminor axis ϵ. Thus in the Hausdorff metric topology $\lim_{\epsilon\to 0} V(\epsilon) = U$, and $\partial R(t, x, \epsilon)$ is smooth with positive Gaussian curvature. From the maximum principle approach define

$$H(t, x, p, v, \epsilon) \equiv p_1 x_2 + p_1 v_1 - p_2 x_1 + p_2 v_2 - 1.$$

Using Lemma 3 one computes the maximizing value of v to be

$$v^*(t, x, p^*) \equiv (\epsilon^2 p_1[\epsilon^2 p_1{}^2 + p_2{}^2]^{-1/2}, p_2(\epsilon^2 p_1{}^2 + p_2{}^2]^{-1/2})$$

from which it follows that $H(t, x, p, v^*(t, x, p); \epsilon)$ which we denote H^* is

$$H^*(t, x, p, \epsilon) \equiv p_1 x_2 - p_2 x_1 + [p_1{}^2 \epsilon^2 + p_2{}^2]^{-1/2} - 1.$$

The associated Hamilton-Jacobi equation as developed in [*6*] is

$$V_t(t, x) + x_2 V_{x_1}(t, x) - x_1 V_{x_2}(t, x) + [\epsilon^2 v_{x_1}^2(t, x) + V_{x_2}^2(t, x)]^{1/2} - 1 = 0.$$

Since the independent variables appear linearly, while the dependent variable has derivatives which appear nonlinearly, the Legendre contact transformation is suggested. Let $V(t, x) = W(t, p) - p \cdot x$. Then $V_t = W_t$, $V_x = -p$, $W_p = x$ and the transformed equation is

$$W_t(t, p) - p_1 W_{p_2}(t, p) + p_2 W_{p_1}(t, p) + [\epsilon^2 p_1{}^2 + p_2{}^2]^{1/2} - 1 = 0.$$

The characteristic equations associated with this linear partial differential equation are $t'(\tau) = 1$, $p_1'(\tau) = p_2(\tau) = -p_1(\tau)$, yielding solutions:

$$t = \gamma + \tau, \qquad p_1 = \alpha \sin(\tau + \beta), \qquad p_2 = \alpha \cos(\tau + \beta)$$

with α, β, γ arbitrary constants. Then

$$\frac{d}{dt} W(t(\tau), p(\tau)) = 1 - [\epsilon^2 p_1{}^2(\tau) + p_2{}^2(\tau)]^{1/2}$$

which, after a slight calculation, gives

$$W(t, p_1, p_2; \delta, \gamma) = t - \gamma + \delta + \int_0^{(\gamma - t)} [\epsilon^2(p_2 \sin\tau + p_1 \cos\tau)^2 + (p_2 \cos\tau - p_1 \sin\tau)^2]^{1/2}\, dt.$$

For a time optimal problem with autonomous system equations and target a point in state space, the constant δ is inconsequential. We consider $\delta = 0$ and omit further reference to it.

By virtue of the transformation, solution trajectories to the system (9) with $v = v^*(t, x, p)$ are given by $x(t; \alpha, \beta, \gamma) = W_p(t, p(t; \alpha, \beta); \gamma)$ or specifically

$$x_1(t; \alpha, \beta, \gamma) = \int_0^{(\gamma-t)} \frac{\alpha\epsilon^2 \sin(2\tau + \beta)\cos\tau - \alpha\cos(2\tau + \beta)\sin\tau}{[\epsilon^2\alpha^2\sin^2(2\tau + \beta) + \alpha^2\cos^2(2\tau + \beta)]^{1/2}}\,d\tau$$

$$x_2(t; \alpha, \beta, \gamma) = \int_0^{(\gamma-t)} \frac{\alpha\epsilon^2 \sin(2\tau + \beta)\sin\tau + \alpha\cos(2\tau + \beta)\cos\tau}{[\epsilon^2\alpha^2\sin^2(2\tau + \beta) + \alpha^2\cos^2(2\tau + \beta)]^{1/2}}\,d\tau \tag{10}$$

These formulas can be interpreted as follows. If we choose $\gamma > 0$ and $t = 0$, $\{x(0; \alpha, \beta, \gamma) : (\alpha, \beta) \in E^2\}$ gives the set of initial points x_0 from which the origin can be reached in time γ by trajectories which satisfy (9) with $v = v^*(t, x, p)$. In particular, it can be shown (via the theory of homogeneous contact transformations) that the jacobian determinant $\partial(x_1, x_2)/\partial(\alpha, \beta)$ is zero, and in this case the set of initial points forms a closed curve in E^2 for each $\gamma > 0$.

To generate a field of extremals (it is to be cautioned that the term extremal is to be taken in the sense of the classical calculus of variations; i.e., not necessarily to infer optimality) choose $\gamma = 0$ and replace t with $-t$ in (10). For each choice of α, β one obtains an extremal which is at the origin at time zero. Varying α, β now gives a field of extremals.

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